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The $(\mathbb{Z}_n, +)$ Group That Build K$(\mathbb{Z}_n)$-Algebras

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Abstrak. Konsep yang diterapkan dalam K-aljabarhampir sama dengan konsep dalam grup. Jika dalam grup terdapat homomorfisma grup, mak dalam K-aljabar terdapat K-homomorfisma. Penelitian ini membahas mengenai struktur dan sifat-sifat yang terkait dengan K-aljabar, K- homomorfisma dan juga membahas tentang suatu grub yang dapat membangun K-aljabar yaitu grup (Z_n,+_n). Penelitian ini menggunakan metode kajian pustaka, dengan cara mengumpulkan berbagai sumber dan teorema- teorema yang mendukung penelitian.

Abstract. The concept which is applied in the K-algebra is similar to the concept of group. If in group there is a group homomorphism,then in K-algebra there is K-homomorphism. This study discusses the structure and properties associated with the K- algebra and K-homomorphism and also discuss about a grub which can build K-algebra that is (Z_n,+_n) group. This research used literature review, by collecting a variety of sources and theorems that support the research.

Keywords: *Group; homomorphism; K-algebra; K-homomorphism.*

1 Introduction

Algebra is a non-empty set that is equipped with one or more binary compositions. Suppose S is a set which is completed with two binary compositions + and *, then S becomes an algebraic structure and is given the notation (S, +, *). Algebraic structures can be classified in general based on their properties, namely groupoid, semigroup, monoid, group. Which can then be grouped again specifically based on more specific properties. Grub theory is one of the abstract algebraic studies that studies the structure of sets, groups used in mathematics and the natural sciences. Let G = <G,*> a group against binary operations *, if e is the identity element with respect to G and for every x,y in G defined the operation x⨀y=x*y^(-1) such that the operation is an operation binary that fulfills these axioms then G will form an algebraic structure that matches K-algebra (Dar and Akram 2014). If in the group we recognize subgroups and group homomorphisms, then in K-algebra we recognize the existence of K-subalgebra, and K-homomorphism. K-algebra is divided into two

2 Material and Method K-Algebras, K-Homomorphism, and B-Algebras

Definition 2.1(Dar and Akram 2014)*Let* $(A;\circledast,*,e)$ *is an algebra defined on the group* $(A,\circledast)$ *in which each of non-identity is not of order 2, then* $(A;\circledast,*,e)$ *is* $K(A)$*--algebra if it satisfies the following properties*

(KA1) $(a*b)*(a*c) = \big(a*(c^{-1}*b^{-1})\big)*a$,

(KA2) $a*(a*b) = (a*b^{-1})*a$,

(KA3) $(a*a) = e$,

(KA4) $(a*e) = a$,

(KA5) $(e*a) = a^{-1}$,

for every $a,b,c \in A$.

If the group $(A,\circledast)$ is commutative group, then for every $x,y,z \in A$

(KA6) $(a*b)*(a*c) = c*b$,

(KA7)$a*(a*b) = b$.

Definition 2.2(Iswati and Suryoto n.d.)*Let* $(A,*,e)$ *be K-algebra, then* $(A,*,e)$ *is commutative if*

$$a*(e*b) = b*(e*a),$$

for every $a,b \in A$.

Definition 2.3(Dar and Akram 2007)*Let* $(A_1,*,e)$ *and* $(A_2,*',e')$ *be K-algebra. A function* f *from* A_1 *to* A_2*, donated by* $f: A_1 \to A_2$ *is called K-homomorphism if for every* $a,b \in A_1$*satistified*

$$f(a*b) = f(a)*'f(b)$$

where $f(a), f(b) \in A_2$.

Definition 2.4*Let* $(A_1,*,e)$ *and* $(A_2,*',e')$ *be K-algebra, then* $Hom(A_1,A_2)$ *is the set all K-homomophism from* A_1 *to* A_2.

Preposition 2.5(Dar and Akram 2007)*Let* $(A_1,*,e)$ *and* $(A_2,*',e')$ *be K-algebra, and let* $f \in Hom(A_1,A_2)$ *then*

$$f(e) = e',$$

$$\left(f(a)\right)^{-1} = f(a^{-1})$$

$$f(e * a) = e' *' f(a),$$

for every $a \in A_1$ *and* $f(a) \in A_2$.

Proof. Let $a \in A_1$ then

1) $f(e) = f(a * a)$ [by KA3]

$= f(a) *' f(a)$

$= e'$

2) $f(a^{-1}) = f(e * a)$ [by KA5]

$= f(e) *' f(a)$

$= e' *' f(a)$ [by Preposition 2.4 number 1]

$= \left(f(a)\right)^{-1}$ [by KA5]

3) $f(e * a) = f(e) *' f(a)$

$= e' *' f(a)$ [by Preposition 2.4 number 1]

Definition 2.6(Neggers and Kim 2002)*A non-empty set Awith the binary operation " $*$ "and constant 0 is B-algebra if it satisfies:*

(BA1)$a * a = 0$,

(BA2) $a * 0 = a$,

(BA3)$(a * b) * c = a * \big(c * (0 * b)\big)$,

for every $a, b, c \in A$.

Proposition 2.7(Wicaksono, Sumanto, and Irawanto 2021)*Let* $(A; \circ, 0)$ *be group with identity element 0. If defined* $a * b = a \circ b^{-1}$ *for every* $a, b \in A$*, then* $(A; *, 0)$ *is B-algebra.*

Proof. Let $a, b, c \in A$ then

1) $a * a = a \circ a^{-1} = 0$

2) $a * 0 = a \circ 0^{-1} = a \circ 0 = a$

3) $(a * b) * c = (a \circ b^{-1}) \circ c^{-1}$

$= a \circ (b^{-1} \circ c^{-1})$

$= a \circ (c \circ b)^{-1}$

$= a * (c \circ b)$

$= a * (c * b^{-1})$

$= a * (c * (0 * b))$

Theorem 2.8(Dar and Akram 2014)*Let* $(A; *, 0)$ *be B-algebra and let* $(A, \circledast)$ *be a group that construct B-algebras. Then K(A)-algebra is also can be built on the group* $(A, \circledast)$.

Proof. Defined$a * b = a \circledast b^{-1}$ for every $x, y \in X$ then based on Preposition 2.7$(A; *, 0)$ is B-algebra.

Let $a, b, c \in A$ then

1) $(a * b) * (a * c) = a * \big((a * c) * (0 * b)\big)$

$= a * (a * ((0 * b) * (0 * c))$ [by BA3]

$= a * \left(a * (b^{-1} \circledast c)\right)$

$= (a * \left(0 * (b^{-1} \circledast c)\right) * a$ [by BA3]

$= (a * (b^{-1} \circledast c)^{-1}) * a$

$= (a * \left((c^{-1} \circledast b)\right) * a$

$= \left(a * (b^{-1} * c^{-1})\right) * a$

2) $a * (a * b)$ $= \left(a * (0 * c)\right) * a$ [by BA3]

$= (a * c^{-1}) * a$

3) $(a * a)$ $= a \circledast a^{-1}$

$= e$

4) $(a * e)$ $= (a \circledast e^{-1})$

$= (a \circledast e)$

5) $(e * a)$ $= (e \circledast a^{-1})$

$= a^{-1}$

So it is proven that K(A)-algebra is also can be built on the group $(A,\circledast)$.

Remark

Function $f: A_1 \to A_2$ is K-homomorphism and also group-homomorphism.

A K-homomorphism f is K-monomorphism, K-epimorphism, dan K-isomorphisms f is one-one, onto, and bijective respectively.

A K-homomorphism f is K-endomorphism if $A_1 = A_2$.

A K-homomorphism f is K-automorphism if $A_1 = A_2$ and f bijective.

Selected solvents during the experiments were utilized based on some reference from previous studies of solvent extraction process or also called with soil washing as summarized at the following table.

3 K(Z_n)-Algebras

Theorem 3.1 Let $(\mathbb{Z}_n, +_n)$ be group with identity element $[0]_n$, if defined binary operations " $*$ " in $\mathbb{Z}_n$ with

$$[a]_n * [b]_n = [a]_n -_n [b]_n,$$

for every $[a]_n, [b]_n \in \mathbb{Z}_n$, then $(\mathbb{Z}_n; +_n, *, [0]_n)$ is K($\mathbb{Z}_n$)-algebra.

Proof. It can be proven that $(\mathbb{Z}_n, +_n)$ is a group that has an identity element $[0]_n$ and defined with

$$\begin{aligned} [a]_n * [b]_n &= [a]_n -_n [b]_n \\ &= [a]_n +_n (-_n[b]_n) \\ &= [a + (-b)]_n \end{aligned}$$

Also for every $[a]_n \in \mathbb{Z}_n$ there exist $[a]_n^{-1} = -_n[a]_n \in \mathbb{Z}_n$ called inverse of $[a]_n$ such that

$$[a]_n +_n (-_n[a]_n) = [0]_n = (-_n[a]_n) +_n [a]_n.$$

Let $[a]_n, [b]_n, [c]_n \in \mathbb{Z}_n$ then

1) $$\begin{aligned} ([a]_n * [b]_n) * ([a]_n * [c]_n) &= ([a]_n -_n [b]_n) -_n ([a]_n -_n [c]_n) \\ &= ([a]_n +_n (-_n[b]_n)) +_n ([c]_n +_n (-_n[a]_n)) \\ &= [a]_n +_n (-_n[b]_n +_n [c]_n) +_n (-_n[a]_n) \\ &= [a]_n -_n (-_n[c]_n +_n [b]_n) -_n [a]_n \\ &= [a]_n -_n (-_n[c]_n -_n (-_n[b]_n)) -_n [a]_n \end{aligned}$$

$= [a]_n * \left([c]_n{}^{-1} * (-_n[b]_n)\right) * [a]_n$

$= [a]_n * \left([c]_n{}^{-1} * [b]_n{}^{-1}\right) * [a]_n$

2) $[a]_n * ([a]_n * [b]_n) = [a]_n -_n ([a]_n -_n [b]_n)$

$= [a]_n +_n ([b]_n +_n (-_n[a]_n))$

$= ([a]_n +_n [b]_n) +_n (-_n[a]_n)$

$= \left([a]_n -_n (-_n[b]_n)\right) -_n [a]_n$

$= ([a]_n * (-_n[b]_n)) * [a]_n$

$= \left([a]_n * [b]_n{}^{-1}\right) * [a]_n$

3) $[a]_n * [a]_n = [a]_n -_n [a]_n$

$= [a]_n +_n (-_n[a]_n)$

$= [a + (-a)]_n$

$= [0]_n$

4) $[a]_n * [0]_n = [a]_n -_n [0]_n$

$= [a]_n +_n (-_n[0]_n)$

$= [a + (-0)]_n$

$= [a]_n$

5) $[0]_n * [a]_n = [0]_n -_n [a]_n$

$= [0]_n +_n (-_n[a]_n)$

$= [0 + (-a)]_n$

$$= [-a]_n$$

$$= -_n[a]_n$$

$$= [a]_n{}^{-1}$$

So based on Definition 2.1 it is proven that $(\mathbb{Z}_n; *, [0]_n)$ is K$(\mathbb{Z}_n)$-algebra.

Theorem 3.2 Let $(\mathbb{Z}_n, +_n)$be group with identity element $[0]_n$, if defined binary operations " $*$ " in $\mathbb{Z}_n$ with

$$[a]_n * [b]_n = [a]_n -_n [b]_n,$$

for every $[a]_n, [b]_n \in \mathbb{Z}_n$, then $(\mathbb{Z}_n; +_n, *, [0]_n)$ is commutativeK$(\mathbb{Z}_n)$-algebra.

Proof. In Theorem 3.1 it is proven that $(\mathbb{Z}_n; +_n, *, [0]_n)$ isK$(\mathbb{Z}_n)$-algebra.

Let $[a]_n, [b]_n \in \mathbb{Z}_n$ then

$$\begin{aligned}
[a]_n * ([0]_n * [b]_n) &= [a]_n -_n ([0]_n -_n [b]_n) \\
&= [a]_n +_n ([b]_n -_n [0]_n) \\
&= [a]_n +_n [b]_n \\
&= [a + b]_n \\
&= [b + a]_n \\
&= [b]_n + [a]_n \\
&= [b]_n -_n (-_n [a]_n) \\
&= [b]_n -_n ([0]_n -_n [a]_n) \\
&= [b]_n * ([0]_n * [a]_n)
\end{aligned}$$

So based on Definition 2.2 it is proven that $(\mathbb{Z}_n; +_n, *, [0]_n)$ is commutative K$(\mathbb{Z}_n)$-algebra.

Theorem 3.3(Wicaksono, Sumanto, and Irawanto 2021)Let $(\mathbb{Z}_n, +_n)$ be group and let function $f: \mathbb{Z}_n \to \mathbb{Z}_n$, then function f is a group homomorphism if and only if it exists $[k]_n \in \mathbb{Z}_n$ so that $f([m]_n) = [k]_n[m]_n$for every $[m]_n \in \mathbb{Z}_n$.

Proof.

(⇐) Suppose there is $[k]_n \in \mathbb{Z}_n$ so that $f([m]_n) = [k]_n[m]_n$. It will be shown f is the group homomorphism.

Let $[a]_n, [b]_n \in \mathbb{Z}_n$ then

$f([a]_n +_n [b]_n) = [k]_n([a]_n +_n [b]_n)$

$= [k]_n[a + b]_n$

$= [k(a + b)]_n$

$= [ka + kb]_n$

$= [ka]_n +_n [kb]_n$

$= [k]_n[a]_n +_n [k]_n[b]_n$

$= f([a]_n) +_n f([b]_n)$

So it is proven f is the group homomorphism.

(⇒) Suppose f is the group homomorphism, then for every $[m]_n \in \mathbb{Z}_n$

$$f([m]_n) = f\left(\underbrace{[1]_n +_n [1]_n +_n \ldots +_n [1]_n}_{m}\right)$$

$$f([m]_n) = \underbrace{f([1]_n) +_n f([1]_n) +_n \ldots +_n f([1]_n)}_{m}$$

$$f([m]_n) = m \times f([1]_n)$$

Suppose $f([1]_n) = [k]_n \in \mathbb{Z}_n$ then

$f([m]) = mf([1]_n)$

$= m[k]_n$

$= [mk]_n$

$= [km]_n$

$= [k]_n[m]_n,$

So if f is a group homomorphism then it is proven there exists $[k]_n \in \mathbb{Z}_n$ so that $f([m]_n) = [k]_n[m]_n$for every $[m]_n \in \mathbb{Z}_n$.

Theorem 3.4 Let $(\mathbb{Z}_n, +_n)$ be group and function $f: \mathbb{Z}_n \to \mathbb{Z}_n$ is group homomorphism, if $(\mathbb{Z}_n; +_n, *, [0]_n)$ is K($\mathbb{Z}_n$)-algebra which is defined with

$$[a]_n * [b]_n = [a]_n -_n [b]_n,$$

for every $[a]_n, [b]_n \in \mathbb{Z}_n$, then $f: \mathbb{Z}_n \to \mathbb{Z}_n$ is also K-homomorphism.

Proof. It is known f is group homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ then based on Theorem 3.2 there exists$[k]_n \in \mathbb{Z}_n$ so that $f([m]_n) = [k]_n[m]_n$for every$[m]_n \in \mathbb{Z}_n$.

Let $[a]_n, [b]_n \in \mathbb{Z}_n$ then

$f([a]_n * [b]_n) = [k]_n([a]_n * [b]_n)$

$= [k]_n([a]_n -_n [b]_n)$

$= [k]_n\big([a]_n +_n (-_n[b]_n)\big)$

$= [k]_n[a + (-b)]_n$

$= [k\big(a + (-b)\big)]_n$

$= [ka + (-kb)]_n$

$= [ka]_n +_n (-_n[kb]_n)$

$= [ka]_n -_n [kb]_n$

$= [k]_n[a]_n -_n [k]_n[b]_n$

$= [k]_n[a]_n * [k]_n[b]_n$

$= f([a]_n) * f([b]_n)$

So it is proven that $f: \mathbb{Z}_n \to \mathbb{Z}_n$ is also K-homomorphism.

Theorem 3.5 Let $(\mathbb{Z}_n; +_n, *, [0]_n)$ be K$(\mathbb{Z}_n)$-algebra and function $f: \mathbb{Z}_n \to \mathbb{Z}_n$ is K-homomorphism. Then

A K-homomorphism f is K-monomorphism, K-epimorphism, dan K-isomorphisms.

A K-homomorphism f is K-endomorphism.

A K-homomorphism f is K-automorphism.

Proof. It is known f is group homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ then based on Theorem 3.2 there exists $[k]_n \in \mathbb{Z}_n$ so that $f([m]_n) = [k]_n[m]_n$ for every $[m]_n \in \mathbb{Z}_n$.

Let $[a]_n, [b]_n \in \mathbb{Z}_n$, Suppose $f([a]_n) = f([b]_n)$ then

$$f([a]_n) = f([b]_n) \quad (\Longrightarrow) f([a]_n) -_n f([b]_n) = [0]_n$$

$$(\Longrightarrow)[k]_n[a]_n -_n [k]_n[b]_n = [0]_n$$

$$(\Longrightarrow)[k]_n([a]_n -_n [b]_n) = [0]_n$$

$$(\Longrightarrow)[a]_n - [b]_n = [0]_n$$

$$(\Longrightarrow)[a]_n = [b]_n$$

So it is proven that f is one-one then f is K-monomorphism.

Because for every $[b]_n = [k]_n[a]_n \in \mathbb{Z}_n$, there exist $[a]_n \in \mathbb{Z}_n$ such that

$$[b]_n = [k]_n[a]_n = f([a]_n).$$

So f is onto then f is K-epimorphism.

Because f is one-one and onto then f is bijective so f is K-isomophism.

Because $f: \mathbb{Z}_n \to \mathbb{Z}_n$ is K-homomorphism then f is K-endomorphism.

4 Conclusion

K-algebra is an algebraic structure that can be generated by a group, because $(\mathbb{Z}_n, +_n)$ is a group with identity element $[0]_n$, so that K-algebra can be build from $(\mathbb{Z}_n, +_n)$ group. Because K$(\mathbb{Z}_n)$-algebra $(\mathbb{Z}_n; +_n *, [0]_n)$ is build from a group $(\mathbb{Z}_n, +_n)$ so that group homomorphism $\varphi: \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ is also K-homomorphism and has several traits.

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