A Note on Stefan Wave Problem

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Abstract. In this paper, the so-called Stefan problem for wave equation is studied. A smooth obstacle placed at one end of a vibrating string causes the effective length of the vibrating string to be changing over time. A straightforward perturbation expansion is used to solve the problem analytically. Some remarks regarding studies of a similar problem by others in the literature are given. Furthermore, we implement numerical method to solve the problem and analyzes the result comprehensively.

Keywords: *finite difference; perturbation solution; Stefan problem; wave equation.*

1 Introduction

Vibrations of a one-dimensional string has been an object of studies since the 19th century. There are many interesting applications of this problem, in particular vibration of string in musical instruments. Various mathematical models and analysis have been developed to understand, modify, and adapt the resulting sound of corresponding instruments. Even though string vibration can be easily modelled by a boundary value problem of wave equation type, a more realistic model requires some more complex settings. One of it is the presence of a smooth bridge on one end of the string which leads to a time-dependent domain, since one of the boundaries is moving. This type of problem is known as the Stefan problem, and it requires different mathematical techniques to construct solutions.

The analysis of the vibration of a string with moving boundary has been done since decades ago. One of the first analysis is done by Balazs [1] which studied wave equation with constantly expanding domain. In the recent development, many studies focused on musical instrument, in which an obstacle or barrier exists which disturb the vibration. Various approaches has been done in studying the vibration, such as modal approach [2], using d'Alembert formula [3], using general integral transform [4]. etc. These vibrations can also be studied from the perspective of the collision between the string and the barrier, as is done by Bilbao using finite difference as a numerical scheme [5] and also by Issanchou et al using non-smooth contact dynamics [6].

Many of previously mentioned studies modelled the obstacle in the middle of the domain, so that problem domain remains fixed. If the obstacle is set at one of the boundaries, then the attachment position itself is varying as the string touches the obstacle. This leads to a Stefan problem. Vyasarayani et al derived and modelled this problem comprehensively in the case of Sitar instrument [7]. The frequencies and modes of the vibration of the same model is then studied by Mandal and Wahi using linearization and boundary immobilization technique [8]. They studied further the mode-locking phenomena that occurred in the problem using method of multiple scales [9]. The problem can be extended to accommodate vibration in extra dimension as doubly curved obstacle is considered. In this case of problem, planar and non-planar motions can be studied using Galerkin projections [10], [11].

In this study, we consider the same model as is derived Vyasaravani et al [7] using somewhat different approaches. We apply immobilization method as is done by Mandal and Wahi but we use different scaling and steps. We show that the transformed nonlinear problem can be written as a family of simple homogeneous linear problem using the perturbation expansion. We also applied numerical computation to the small order part problem without Galerkin projections.

2 Problem Formulation

Consider a thin string with length L, tension T, and length-density ρ. The string is fixed at one end: $X = L$. At the left boundary, a small bridge with parabolic-shaped which defined by $Y_B(X) = A_p X(B - X)$ for some given value of $A_p > 0$ and $B \in (0, L)$, is placed. This bridge is touched by the string as the string displaces, shifting the attachment point over time. This gives rise to a moving boundary problem. The displacement of the string vibrates over time, and it is governed by following system.

$$TY_{XX} - \rho Y_{\theta\theta} = 0, \quad \Gamma(\theta) \leq X \leq L \tag{1}$$
$$Y(\Gamma, \theta) = Y_B(\Gamma) = A_p \Gamma(B - \Gamma) \tag{2}$$
$$Y(L, \theta) = H_r \tag{3}$$
$$Y_X(L, \theta) = \frac{dY_B}{dX}(\Gamma) = A_p(B - 2\Gamma) \tag{4}$$

We apply scaling transformation to obtain dimensionless system by the following transformations

$$\bar{x} = \frac{X}{L}, \;\; \bar{y}(\bar{x}, \bar{\tau}) = \frac{Y(X,\theta)}{A_p L^2}, \; \bar{\tau} = \theta \sqrt{\frac{T}{\rho L^2}}, \;\; \bar{\gamma} = \frac{\Gamma}{L}, \;\; b = \frac{B}{L}, \; h = \frac{H_r}{A_p L^2} \tag{5}$$

We use a different scaling transformation in contrast with the those in [7] and [8]. The resulting system then depends only on two parameters, i.e., b and h. Keeping the parameter A_p as a controlled parameter might give a misleading reconstruction of real phenomenon. This is clear from the previous studies where predefined and scaled A_p when implementing the simulation produce a high gap difference between the peak left obstacle and the right attachment. This is not realistic in any string musical instrument. By setting different scaling length in X and Y coordinate, the height of the obstacle is normalized, thus high gap with the right attachment is avoided. Substituting these to the system gives us

$$\bar{y}_{\bar{x}\bar{x}} - \bar{y}_{\bar{\tau}\bar{\tau}} = 0, \quad \bar{\gamma}(\bar{\tau}) \leq \bar{x} \leq 1 \tag{6}$$
$$\bar{y}(\bar{\gamma}(\bar{\tau}), \bar{\tau}) = \bar{\gamma}(b - \bar{\gamma}) \tag{7}$$
$$\bar{y}(1, \bar{\tau}) = h \tag{8}$$
$$\bar{y}_{\bar{x}}(1, \bar{\tau}) = b - 2\bar{\gamma} \tag{9}$$

We define further initial conditions for this system as follows.

$$\bar{y}(\bar{x}, 0) = \bar{f}(\bar{x}) \tag{10}$$
$$\bar{y}_{\bar{\tau}}(\bar{x}, 0) = \bar{g}(\bar{x}) \tag{11}$$

We can derive further additional initial conditions for γ, i.e, $\gamma(0) = P$ and $\dot{\gamma}(0) = Q$, where P is a solution of $P(b - P) = \bar{f}(P)$ and $Q = \frac{\bar{g}(\alpha)}{b-2\alpha}$.

One of the difficulties in studying of moving boundary problem is the nonfixed domain with respect to time. This issue can be handled by applying boundary immobilization or boundary fixing technique using the following transformation

$$x = \frac{\bar{x}-\bar{\gamma}}{1-\bar{\gamma}}, \;\; y(x,\tau) = \bar{y}(\bar{x},\bar{\tau}), \;\; \tau = \tau(\bar{\tau}), \;\; \gamma(\tau) = \bar{\gamma}(\bar{\tau}) \tag{12}$$

Here, we let the time scaling to be determined later. This transformation turns the system to

$$\left[1 - \left((x-1)\dot{\gamma}\tau_{\bar{\tau}}\right)^2\right] y_{xx} = (1-\gamma)^2 \tau_{\bar{\tau}}^2 y_{\tau\tau} + 2(x-1)(1-\gamma)\dot{\gamma}\tau_{\bar{\tau}}^2 y_{x\tau}$$
$$+(x-1)[2\dot{\gamma}^2 + (1-\gamma)\ddot{\gamma}]\tau_{\bar{\tau}}^2 y_x + (1-\gamma)^2 \tau_{\bar{\tau}\bar{\tau}} y_\tau \tag{13}$$
$$y(0,\tau) = \gamma(b-\gamma) \tag{14}$$
$$y(1,\tau) = h \tag{15}$$
$$y_x(0,\tau) = (b-2\gamma)(1-\gamma) \tag{16}$$

where the domain now is fixed in interval $(0,1)$.

3 Analytical Solution

At this section, we first assume that the string is perturbed initially with small displacement, i.e., $y(x,0) = \varepsilon f(x)$ and $y_\tau(x,0) = \varepsilon g(x)$. We define the perturbation expansion for y and γ as follows.

$$y(x,\tau) = z(x) + \varepsilon v(x,\tau) + \mathcal{O}(\varepsilon^2) \tag{17}$$
$$\gamma(\tau) = \gamma_0 + \varepsilon s(\tau) + \mathcal{O}(\varepsilon^2) \tag{18}$$

Substituting these to eq. (13)-(16) yields the following two systems of equations,

$$z'' = 0 \tag{19}$$
$$z(0) = \gamma_0(b - \gamma_0) \tag{20}$$
$$z(1) = h \tag{21}$$
$$z'(0) = (b - 2\gamma_0)(1 - \gamma_0) \tag{22}$$

for order $\mathcal{O}(1)$ and

$$(1-x)(1-\gamma_0)\ddot{s}\tau_{\bar{\tau}}^2 z' + v_{xx}$$
$$= (1-\gamma_0)^2\tau_{\bar{\tau}}^2 v_{\tau\tau} + (1-\gamma_0)^2\tau_{\bar{\tau}\bar{\tau}} v_\tau \tag{23}$$
$$v(0,\tau) = s(b - 2\gamma_0) \tag{24}$$
$$v(1,\tau) = 0 \tag{25}$$
$$v_x(0,\tau) = -s(b - 4\gamma_0 + 2) \tag{26}$$

for order $\mathcal{O}(\varepsilon)$. Here the "prime" denotes the derivative with respect to time. The corresponding intial condition for v reads, $v(x,0) = f(x)$ and $v_\tau(x,0) = g(x)$. On can easily solve system (19)-(21), i.e.

$$z\,(x) = hx + \gamma_0(b - \gamma_0)(1 - x) \tag{27}$$

Extra condition (22) gives us the relation $\gamma_0 = 1 - \sqrt{1 + h - b}$. Taking the next order into consideration, observe first that by defining time transformation as $\bar{\tau} = \tau(1 - \gamma_0)$, eq. (23) is reduced to

$$v_{xx} = v_{\tau\tau} - \frac{(1-x)\ddot{s}}{1-\gamma_0} z' \tag{28}$$

Next, by defining shifting transformation

$$w(x,\tau) = v(x,\tau) - (b - 2\gamma_0)(1 - x)s(\tau) \tag{29}$$

a homogeneous wave problem is obtained,

$$w_{\tau\tau} = w_{xx} \tag{30}$$

$$w(0,\tau) = w(1,\tau) = 0 \tag{32}$$

which can be solved to

$$w(x,\tau) = \sum_{n=}^{\infty} \phi_n(\tau)\psi_n(x) \tag{33}$$

where $\phi_n(\tau) = A_n \sin(\omega_n \tau) + B_n \cos(\omega_n \tau)$ and $\psi_n(x) = \sin(\omega_n x)$ is corresponding eigenfunction of eigenvalue $\omega_n = n\pi$. Constants A_n and B_n are computed using initial conditions of v, as following

$$A_n = -\frac{2}{\omega_n^2}[g(0) - \omega_n \langle g, \psi \rangle] \tag{34}$$

$$B_n = -\frac{2}{\omega_n}[f(0) - \omega_n \langle f, \psi \rangle] \tag{35}$$

where $\langle \_, \_ \rangle$ is inner product operator in $L^2[0,1]$ space. Additional condition of eq. (26) can be used to obtain explicit solution of s as follows

$$s(\tau) = \sum_{n=1}^{\infty} \frac{\omega_n \phi_n(\tau)}{2(\gamma_0 - 1)} \tag{36}$$

To conclude, the complete solution for y is

$$y(x, \tau) = z(x) + \varepsilon \sum_{n=1}^{\infty} \phi_n(\tau) \left[\psi_n(x) - \frac{\omega_n z'(x)(1-x)}{(1-\gamma_0)^2}\right] + \mathcal{O}(\varepsilon^2) \tag{37}$$

This result is the same as is obtained in [8]. However, in this paper we present a rigorous mathematical derivation rather than assuming the solution as is done in [8]. Moreover, in [8], ϕ_n is set directly with some set of values, separating the solution context from initial conditions of the string.

Given initial conditions $f(x)$ and $g(x)$ above equation can be computed over time τ. However, defining the right $f(x)$ or $\bar{f}(\bar{x})$ is required to have a good solution, otherwise there will be a discontinuity occurring in initial time steps. To be specific, f should be tangent to the surface of the obstacle at $x = 0$.

4 Numerical Solution

Let us now look at a numerical approach for solving the problem. We start with a system with the stationary solution has already been removed (28) with a particular boundary condition. Let Δt and Δx are lengths of small interval slices for t and x. We denote first $v_i^n = v(i\Delta x, n\Delta t)$ and $s^n = s(n\Delta t)$. Using central difference scheme for differential approximation, we obtain

$$v_i^{n+1} = V_i^n - v_i^{n-1} - C_i(s^{n+1} - 2s^n + s^{n-1}) \tag{38}$$

Where $V_i^n = 2v_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 (v_{i+1}^n - 2v_i^n + v_{i-1}^n)$ and $C_i = (i\Delta x - 1)(b - 2\gamma_0)$. This schema should be computed using boundary condition $v_0^n = s^n(b - 2\gamma_0)$ and $v_{N_x}^n = 0$ where N_x is the number of spatial grids. The tricky part of moving boundary problem is that we have two unknowns that must be solved simultaneously. In this case, value of s is needed to compute (38) each time step. Recall that we have an additional boundary condition (26), i.e.

$$\frac{v_i^{n+1} - v_0^{n+1}}{\Delta x} = -s^{n+1}(b + 2 - 4\gamma_0) \tag{39}$$

Substituting this condition to schema (38) gives us

$$s^{n+1} = \frac{1}{2(\gamma_0 - 1)\Delta x}[V_1^n - v_1^{n-1} + C_1(2s^n - s^{n-1}) \tag{40}$$

In addition to initial condition for v, i.e., $v_i^0 = f(i\Delta x)$, we also need initial condition for s to compute the values initially. In that case, it is important to be noted that assumption of small initial string displacement implies also small initial conditions for γ, thus we write $P - \gamma_0 = \varepsilon p$ and $Q = \varepsilon q$. Thus, we have that $s^0 = p$. However, both (38) and (40) need two time-steps backward to be computed. In that case, we use other initial conditions

$$\frac{v_i^1 - v_i^{-1}}{2\Delta t} = (1 - \gamma_0)g(i\Delta x) \quad \text{and} \quad \frac{s^1 - s^{-1}}{2\Delta t} = q \tag{41}$$

which gives us explicit form of v and s at time step 1, i.e.

$$v_i^1 = \frac{1}{2}V_i^0 + \Delta t(1 - \gamma_0)g(i\Delta x) - C_i(s^1 - p - q\Delta t) \tag{42}$$

$$s^1 = \frac{1}{2(\gamma_0 - 1)\Delta x}\left[\frac{1}{2}V_i^0 + \Delta t(1 - \gamma_0)g(\Delta x) + C_1(p + q\Delta t)\right] \tag{43}$$

This completes the schema.

5 Result and Discussion

In this section, we will discuss some computational results and simulation of the problem. We use initial condition as $f(x) = \sin(\pi x)$ and $g(x) = 0$, which represents smooth idle displacement. To be noted that this initial condition does not meet tangential requirements with the obstacle. We will see that even though a discontinuity occurs initially, it does not affect the dynamics afterwards.

We set first $\varepsilon = 0.01$. Value of ε needs to be this small due the scale of variables and parameters involved in the problem. Observe that in our scaled model, value of b should be less than 1. Even in more realistic setup, b should be far less than 1, like 0.1. The height of the obstacle thus become very small as it equals $\frac{b^2}{4}$. This then restrict h to be around that value. Even though mathematically value of h is arbitrary. It won't be realistic if gap between h and the height of the obstacle is too large. In this setting, the vibration will be very small, otherwise the lower amplitude will be too negative, which should be avoided for now, due to practicality and possible violation of tangentiality of the string with the obstacle. To illustrate this point, let $b = 0.1$ and $h = 0.01$. This setup is shown in Fig. 1.

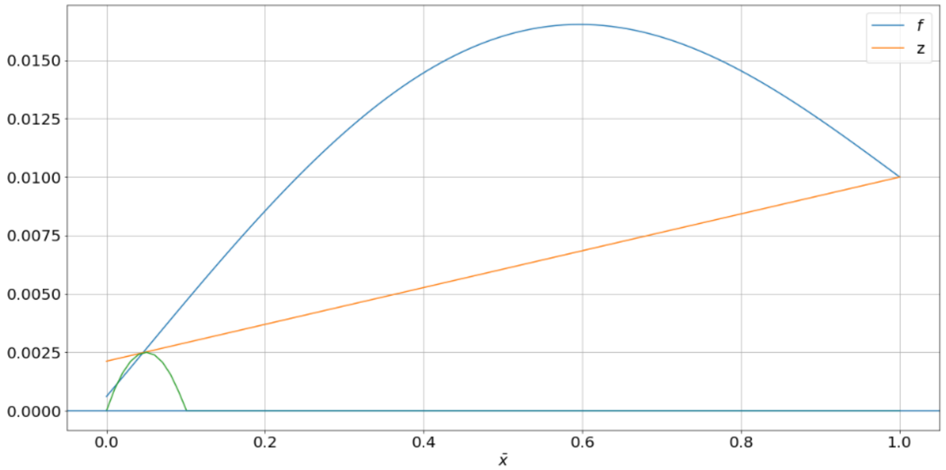

Figure 1 Initial setup of the problem in the original domain $\bar{x}$.

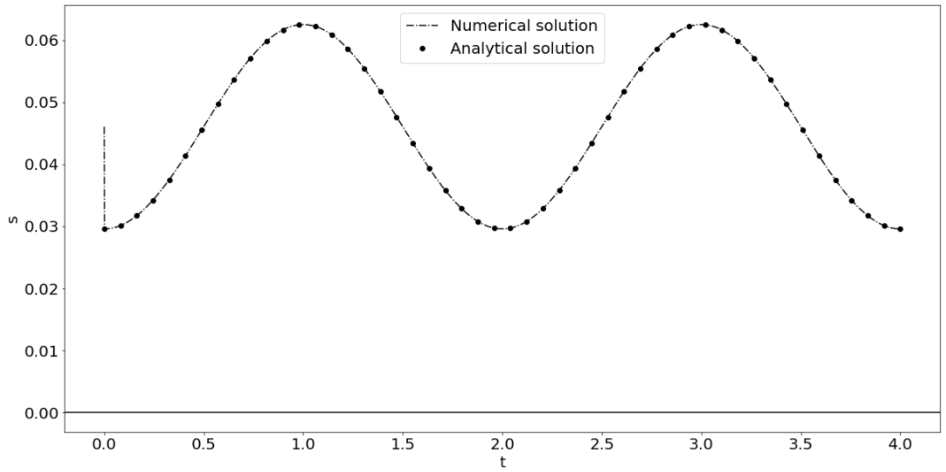

Figure 2 Moving boundary profile $s(t)$ for initial condition $f(x) = \sin(\pi x)$.

We can see even with $\varepsilon = 0.01$, the initial displacement is very large. Mandal and Wahi did mention about that using different length scale for X and Y is to avoid rounding error due to small number computation. However, the value is still large enough to obtain accurate solution numerically. If we simulate the moving boundary s with this setup using both analytical solution (36) and the numerical schema, we obtain a profile shown in Fig. 2.

The solutions agree with negligible difference. A jump at the initial time steps in the numerical result is caused by the initial conditions f that is not tangent with the obstacle. If we look closely at the model setup, as shown in Fig. 3, the initial condition f touches the obstacle improperly at $s(0) = \gamma_0$. Given the slope of the tangent line of f at $s = \gamma_0$, the point where the tangent line of the obstacle has the same slope is around 0.03. In the first time step of numerical integration, this improper condition is corrected. This jump does not happen in analytical solution because it does not require the exact value of $s(0)$.

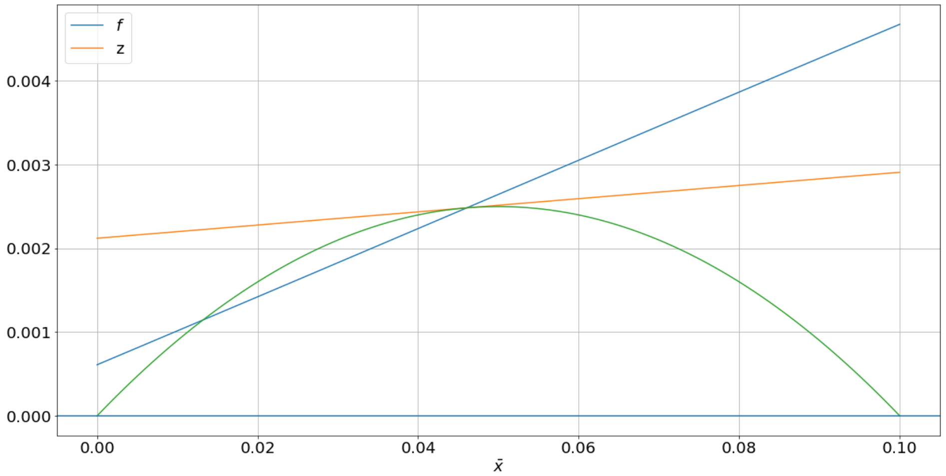

Figure 3 Closer look of the initial setup of the problem. The initial condition f is traversing the obstacle.

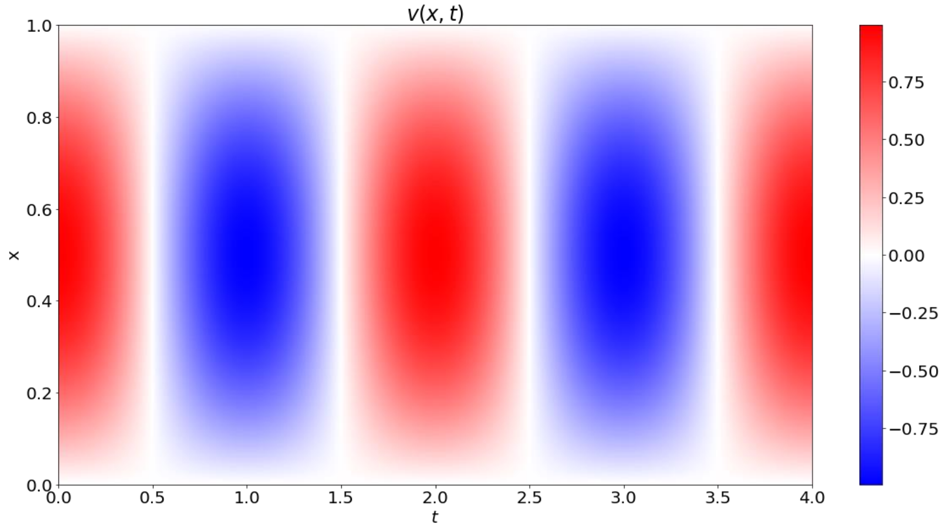

Figure 4 Map of the solution v in spatial and temporal domain using numerical approach.

The profile of moving boundary forms sinusoidal wave as the attachment point is moving back and forth around stationary point γ_0. If we map the solution of v, as shown in Fig. 4, we can see a perfect wave is formed. If we look at the explicit form, eq. (29), there should be a disturbance term caused by the nonlinearity of the moving boundary. However, the term is very small that it barely affects the wave profile.

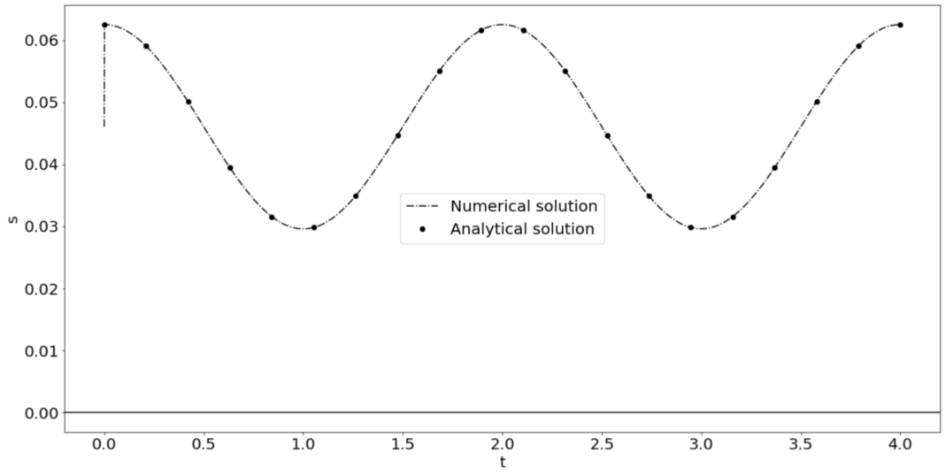

Figure 5 Moving boundary profile $s(t)$ for initial condition $f(x) = -\sin(\pi x)$.

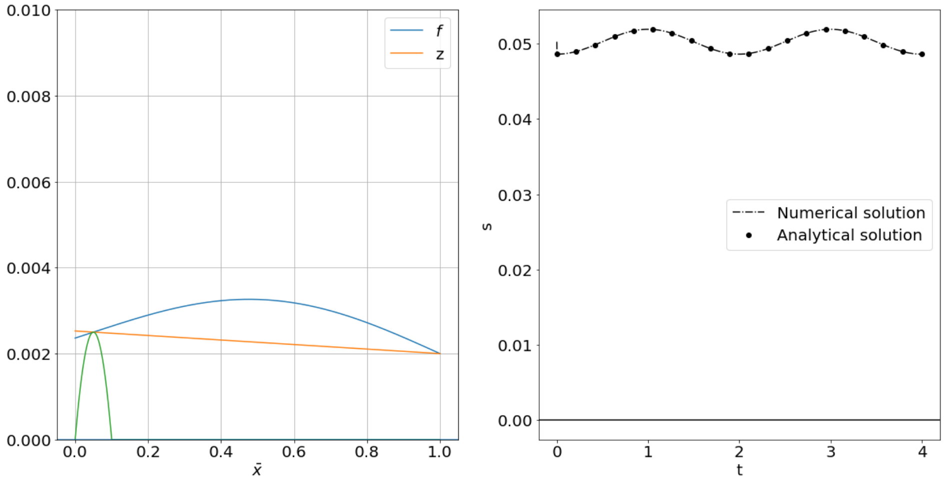

Figure 6 Problem setup (left) and the moving boundary profile (right) with $\varepsilon = 0.001$, $b = 0.1$, and $h = 0.002$.

In previous case, as shown in Fig. 2, the moving boundary profile immediately starts from the left of stationary point γ_0. Positive derivative of the initial condition at the left boundary causes the attachment point shifts to the left as the string is lifted up a bit. On the other hand, if we set the initial condition to have negative derivative at the left boundary, e.g.., $f(x) = -\sin(\pi x)$, the attachment point shifts to the right, as shown in Fig. 5. For another case, if we take smaller value of ε for the sake of realistic model, no computational issue occurs as it is still be computable without experiencing underflow. In Fig. 6.

6 Conclusion

Stefan problem for wave equation has been studied. Straightforward perturbation solution is implemented using some set of nondimensionalization scaling. This work presents some remarks and notes to the solution, especially in matter of scales of the variables as it affects the interpretability of the problem with the real phenomenon. A numerical schema is applied to the problem and gives close results with the analytical solution. More generic Stefan problem for wave equation is suggested for future research.

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