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Correction to the Inflationary Power Spectrum from Spatial Curvature

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Abstract. In this work, we calculate the corrections to the inflationary power spectrum of primordial curvature perturbations arising from small initial spatial curvature prior to inflation. The universe is described by the Friedman-Lemaître-Robertson-Walker (FLRW) model with spatial curvature and undergoing single-field slow-roll inflation. The presence of spatial curvature introduces additional terms to the second-order action of perturbations, which are proportional to the spatial curvature. We specifically focus our investigation on spatial curvature that is of the same order as the slow-roll parameter. The contribution of these additional terms to the power spectrum is evaluated perturbatively around the usual free part of the action in flat space. The Bunch-Davies boundary condition is applied to normalize the mode function. We find that the corrections are proportional to the spatial curvature for the short-wavelength modes. However, these corrections can be enhanced when the largest observable scale is comparable to the curvature scale.

Keywords: *cosmic microwave background; cosmological perturbation theory; inflation; power spectrum; spatial curvature.*

1 Introduction

The understanding of the early universe and its evolution has been revolutionized by the inflationary paradigm, which provides an elegant solution to the problems of the standard Big Bang cosmology [1][2]. Inflation posits a phase of accelerated expansion in the very early universe, during which quantum fluctuations are stretched to cosmological scales, serving as the seeds for the large-scale structure observed in the universe today. The power spectrum of these primordial curvature perturbations is a fundamental quantity that characterizes the statistical properties of these initial fluctuations and plays a crucial role in the predictions of cosmological observables.

While the inflationary scenario has been successful in explaining various aspects of the observed universe, it is important to refine our theoretical understanding by considering possible corrections to the standard predictions. The investigation of such corrections is of great interest as it can provide insights into the underlying physics of the early universe and potentially distinguish between numerous different proposed inflationary models (for example, [3] and [4]). In particular, the presence of spatial curvature, although constrained to be small by observations, can introduce additional effects that modify the scalar power spectrum, e.g., for open [5] and closed [6] universe.

One of the cosmological problems solved by inflation is the flatness problem. The exponential expansion of space during inflation dilutes away the initial curvature, making our universe since then close to spatially flat. According to the final results of the *Planck* mission released in 2018 [7], the constraint of the curvature density parameter Ω_K from *Planck TT, TE, EE + lowE + lensing + BAO* gives

$$\Omega_K = 0.0007 \pm 0.0037, \qquad (95\% \text{ C.L.})$$

This suggests a universe that is in line with being flat but with a slight uncertainty that favors an open universe. The statistical significance and uncertainty of these observational results remain a topic of debate. Recent studies by Di Valentino et al. [8] and Handley in [10] have shown interest in favoring a closed universe as a solution to tensions in the Planck data. However, it is not universally accepted, as there are also opposing views, for example, Efstathiou and Gratton in [11] support the flat universe. Furthermore, the nucleation of a single bubble can also result in an open universe, as demonstrated by Bucher et al. [12].

Despite the acclaimed approximate flatness, inflation by itself does not provide any clue about the spatial curvature. Even if exponential expansion of space during inflation washes away the initial curvature, however, traces of spatial curvature may not be lost forever. It might be preserved in the cosmic microwave background (CMB) as the relevant modes leave the horizon. There exists a possibility in which the primordial curvature leaves an imprint on the CMB. For instance, if the inflation period did not last long enough to produce adequate *e*-foldings, the currently observable scales might have exited the horizon with spatial slices exhibiting noticeable deviation from flat geometry, as shown by Masso *et al.* in [13].

This paper aims to determine the observational implications of a scenario in which spatial curvature is present alongside inflation. We study the correction to the power spectrum from single-field slow-roll inflation, assuming that inflation began while the curvature was of the same order as the slow-roll parameters. To obtain this correction, we employ a perturbative approach similar to the method used for deriving non-Gaussianity, utilizing the in-in formalism as firstly utilized by Maldacena in [18]. Although the power spectrum is not a reliable indicator of initial conditions with curvature unless its spectral running can be determined with high accuracy, this work provides another way in which the power spectrum can be modified because of the spatial curvature, which has been previously calculated in various ways, e.g., analytically using modified mode functions by Clunan and Seery in [14] and Masso *et al.* in [13], and numerically by Handley in [10]. Generally, the corrections are beyond the detection limit of current cosmic microwave background experiments. The parameter range for Ω_K could be determined with greater accuracy by the continued improvement of future observational techniques, for example, the polarimetric survey *PICO* [21], the high-redshift 21-cm survey with *SKA* [22], and the large-scale structure survey *Euclid* [23].

This paper is organized as follows. In Section 2, we discuss the derivation of the quadratic action, starting with introducing the action and the coordinates used, followed by deriving the background equation, and lastly writing the final form of the action that is up to the first order in the slow-roll parameter. In Section 3, we evaluate the two-point correlation function using in-in formalism up to the first order of perturbation. Finally, in Section 4, we present our conclusions based on the findings of our study. Throughout this paper, we adopt the geometrized units where $c = \hbar = 1$ and the reduced Planck mass is $M_\mathrm{P} = (8\pi G)^{1/2} = 1$.

2 The Quadratic Action

In this section, we provide a review of the quadratic actions for scalar perturbations in the spatially curved background. The action has been previously derived in the literature, with the first derivation being by Clunan and Seery in [14].

The inflation model is described by the action of a scalar field $\phi(t, \vec{x})$ (called inflaton) minimally coupled to the gravity,

$$S = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g}\left[R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\right], \quad (1)$$

where g is the metric determinant, R is the Ricci scalar, $V(\phi)$ is the inflaton potential.

The fluctuations of the metric are evaluated within the framework of ADM formalism. We start by writing the perturbed line element as

$$\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + h_{ij}\left(N^i\mathrm{d}t + \mathrm{d}x^i\right)\left(N^j\mathrm{d}t + \mathrm{d}x^j\right), \quad (2)$$

where $N \equiv N(t, \vec{x})$ is the lapse function, $N^i \equiv N^i(t, \vec{x})$ is the shift vector, and $h_{ij} \equiv h_{ij}(t, \vec{x})$ is the induced metric on three-dimensional hypersurfaces of constant time t. The geometry of the spatial slices is described by the intrinsic curvature, $R_{ij}^{(3)}$, the Ricci tensor of the induced metric, and by the extrinsic curvature K_{ij} that reads

$$K_{ij} \equiv \frac{1}{2N}\left(h'_{ij} - \nabla_i N_j - \nabla_j N_i\right) \equiv \frac{1}{N}E_{ij}. \tag{3}$$

The four-dimensional spacetime Ricci scalar R can be written in terms of the three-dimensional Ricci scalar $R^{(3)}$ and the extrinsic curvature K_{ij} as

$$R = R^{(3)} + N^{-2}\left(E^{ij}E_{ij} - E^2\right), \tag{4}$$

where indices are raised with h^{ij}, and $E \equiv h^{ij}E_{ij}$.

The action function can now be written as

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{h}N\left[R^{(3)} - 2V + N^{-2}\left(E^{ij}E_{ij} - E^2\right) + N^{-2}\left(\phi' - N^i\partial_i\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi\right]. \tag{5}$$

Note that there are no time derivatives of N and N_i appearing in the action. Therefore, both are non-dynamical fields that will be fixed by constraint equations obtained by varying the action with respect to N and N^i. These yield

$$R^{(3)} - 2V - h^{ij}\partial_i\phi\partial_j\phi - N^2\left[E_{ij}E^{ij} - E^2 - \left(\dot{\phi} - N^i\partial_i\phi\right)^2\right] = 0 \tag{6}$$

$$\nabla_i\left[N^{-1}\left(E_j^i - E\delta_j^i\right)\right] = 0. \tag{7}$$

These equations are called the Hamiltonian constraint and the momentum constraint, respectively.

The background metric that we are working with is the general Friedmann–Lemaître–Robertson–Walker (FLRW**)** metric,

$$ds^2 = -dt^2 + a^2(t)\gamma^{ij}dx_i dx_j = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right), \tag{8}$$

where K is the spatial curvature ($K > 0$ for closed universe, $K = 0$ for flat universe, and $K < 0$ for open universe), and γ_{ij} called the induced metric, that is, the comoving spatial part of FLRW metric. To obtain the background equation, set $N = 1$ in constraint equations, which corresponds to choosing the time coordinate to cosmic time,

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3K}{a^2}, \tag{9}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 + \frac{K}{a^2}, \tag{10}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{11}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

To produce a sufficient amount of expansion, the scalar field typically must be in a slow-roll phase where $\dot{\phi}$ is small and is maintained to be small over a sufficiently long period. These conditions can be put precisely by requiring the following dimensionless parameters to be small,

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}, \qquad \delta = \frac{1}{H}\frac{\ddot{\phi}}{\dot{\phi}}. \tag{12}$$

Note that if spatial curvature exists, this definition of ϵ is different from the usual definition for the flat case where $\epsilon = -\dot{H}/H^2 = \dot{\phi}^2/2H^2$. In spatially curved universe, we denote $\epsilon_H = -\dot{H}/H^2$ with $\dot{H}$ given by Eq. (10) and ϵ as defined in Eq. (12). These parameters will usually be treated as though they were constants.

Now we perturb the metric. We only consider the scalar perturbation, neglecting tensor and vector perturbations for the reason that they are decoupled from the scalar fluctuations at linear order. We use the spatially flat gauge to fix time and space reparameterizations. In this gauge, the induced metric is taken to be unperturbed, so the perturbation is shifted into the scalar field,

$$N = 1 + \delta N(t, \vec{x}), \qquad N^i = \gamma^{ij}\partial_i \chi(t, \vec{x}), \qquad h_{ij} = a(t)\gamma_{ij}, \tag{13}$$

$$\phi(t, \vec{x}) = \phi_0(t) + \varphi(t, \vec{x}). \tag{14}$$

where δN and χ represent the first order expansion of lapse function and shift vector, respectively. To simplify the notation, we adopt this notation: $\phi_0(t) \rightarrow \phi(t)$, the subscript '0' will be dropped.

To expand the action up to $\mathcal{O}(\varphi^2)$ (or $\mathcal{O}(\varphi^3)$) we only need to evaluate the constraint equations up to $\mathcal{O}(\varphi)$ [18]. The Hamiltonian and the momentum constraint

$$0 = -2\frac{dV}{d\phi}\varphi + 2\delta N\left(-6H^2 + \dot{\phi}^2\right) - 4H\partial^2\chi - 2\dot{\phi}\dot{\varphi}, \tag{15}$$

$$0 = \partial_i\left[2H\delta N - 2K\chi - \dot{\phi}\varphi\right] = 0. \tag{16}$$

From these equations, we obtain δN and χ up to $\mathcal{O}(\varphi)$, respectively, as

$$\delta N = \frac{\dot{\phi}}{2H}\varphi + \frac{K}{H}\chi = \frac{1}{\sqrt{2}}\sqrt{\epsilon}\varphi + K\frac{\chi}{H}, \tag{17}$$

$$\chi = \frac{H}{\sqrt{2}}\sqrt{\epsilon}(\partial^2 + (3-\epsilon)K)^{-1}\left[(\delta+\epsilon)\varphi - \frac{\dot{\varphi}}{H}\right]. \tag{18}$$

Note that these quantities are at most $\mathcal{O}(\epsilon^{1/2})$.

By substituting Eqn. (17) and (18) into Eqn. (5) and expanding the terms up to $\mathcal{O}(\varphi^2)$, we obtain the second-order action as

$$\begin{aligned} S_2 = \frac{1}{2}\int \mathrm{d}t\, \mathrm{d}^3\mathrm{x}\; a^3\sqrt{\gamma}\Bigg[&\left\{-\left(\frac{d^2V}{d\phi^2}\right) - \frac{\dot{\phi}}{H}\left(\frac{dV}{d\phi}\right)\right\}\varphi^2 - \frac{1}{a^2}\partial_i\varphi\partial^i\varphi \\ &+ 2K\partial_i\chi\partial^i\chi + \dot{\varphi}^2 - 2\frac{\dot{\phi}^2}{H}\varphi\dot{\varphi} - 2\frac{K}{H}\left(\frac{dV}{d\phi}\right)\chi\varphi \\ &- \frac{K\dot{\phi}}{H}\chi\dot{\varphi} + \left(-6 + \frac{\dot{\phi}^2}{H^2}\right)\left(\frac{\dot{\phi}^2}{4}\varphi^2 + K\dot{\phi}\chi\varphi + \chi^2\right)\Bigg]. \end{aligned} \tag{19}$$

This action was derived first by Clunan & Seery in [14] and then in Sugimura & Komatsu in [15]. The leading terms of the Lagrangian are

$$S_{2,0} \approx \frac{1}{2}\int \mathrm{d}t\, \mathrm{d}^3x\, a^3\sqrt{\gamma}\left[\dot{\varphi}^2 - \frac{1}{2a^2}\partial_i\varphi\partial^i\varphi\right] \subset S_2, \tag{20}$$

where the sub-leading terms in the slow-roll parameters and spatial curvature K are neglected in this expression.

Next, we proceed to simplify the action in Eq. (19). The terms containing potential derivatives can be expressed in terms of slow roll parameters using the Klein-Gordon equation, Eq. (11). The term $2a^3\,\dot{\phi}^2\varphi\dot{\varphi}/H$ in Eq. (19) can be converted into mass term, along with a total time derivative that is non-dynamical. In this work, we will only work up to the first order in slow-roll, $\mathcal{O}(\epsilon,\delta)$, so we are left with

$$\begin{aligned} S_2 = \frac{1}{2}\int dt\, d^3x\, a^3\sqrt{\gamma}&\left[\dot{\varphi}^2 - \frac{1}{a^2}\partial_i\varphi\partial^i\varphi + 3H^2[2\epsilon+\delta]\varphi^2\right] \\ &+ \int dt\, d^3x\, \mathcal{L}_{2,\mathrm{K}} \end{aligned} \tag{21}$$

$$\mathcal{L}_{2,\mathrm{K}} = a^3\sqrt{\gamma}\left(-3K^2\chi^2 + K\partial_i\chi\partial^i\chi + \frac{3}{2}\frac{K}{a^2}\varphi^2\right),$$

where

$$\chi = -\frac{1}{\sqrt{2}}\sqrt{\epsilon}(\partial^2 + 3K)^{-1}(\dot{\varphi}). \tag{22}$$

This action is evaluated perturbatively in the next section.

3 The Computation of Two-Point Correlation Function

In this section, we evaluate the power spectrum resulting from inflation in a non-flat background using sub-curvature approximation [15], that is, only considering the modes that are much larger than the curvature scale. This approximation is reasonable because we have not yet detected any strong spatial curvature within our current observable horizon. Furthermore, we also make an approximation regarding the value of K; it is taken to be $\mathcal{O}(\epsilon)$ at the beginning of the inflation. This allows us to focus on calculating the correction to the power spectrum from the additional terms from the Lagrangian, rather than from the mode function in the curved spatial section. The mode function used here is the same as in the flat case.

We begin by reviewing the quantization of the inflaton perturbation and deriving the mode function, both of which are needed when deriving the two-point correlation function at different spacetime points, called the Wightman function. Here, we are closely following Hael in [16] and Baumann in [17].

The scalar field perturbation φ can be transformed into Fourier modes,

$$\varphi(\eta, \vec{x}) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\left\{\varphi_k(\eta)e^{i\vec{k}\cdot\vec{x}}a_{\vec{k}} + \varphi_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}}a_{\vec{k}}^\dagger\right\}, \tag{23}$$

where $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$ is creation and annihilation of the scalar field that satisfy $a_{\vec{k}}|0\rangle = 0$, and $\varphi_k(\eta)$ is the mode function. Note that the mode function only depends on the magnitude of the wavevector, $k = |\vec{k}|$, where η is the conformal time. The conformal time can be conveniently expressed in de Sitter forms as follows,

$$d\eta = \frac{dt}{a} = e^{-Ht}dt \;\Rightarrow\; \eta = -\frac{1}{aH}. \tag{24}$$

From now we are working with conformal time instead of cosmic time, for convenience. It is also convenient to define the canonically normalized field $v \equiv a\varphi$, so that the free part of the action, Eq. (20), becomes

$$S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3x\left[\dot{v}^2 - (\partial_i v)^2 + \frac{\ddot{a}}{a}v^2\right]. \tag{25}$$

This action implies the following equation of motion for the Fourier modes of the field,

$$\ddot{v}_{\rm k} + \left(k^2 - \frac{\ddot{a}}{a}\right) v_{\rm k} = 0. \tag{26}$$

This equation is called the Mukhanov-Sasaki equation. The solution of this equation is

$$v_k(\eta) = \frac{i}{\sqrt{2k^3}} \frac{1 + ik\eta}{-\eta} e^{-ik\eta}. \tag{27}$$

The expression for $\varphi_k(t)$ is then

$$\varphi_k(\eta) = \frac{v_k(\eta)}{a(\eta)} = \frac{H}{\sqrt{2k^3}} (1 + ik\eta) e^{-ik\eta}. \tag{28}$$

The Wightman function is defined as

$$\begin{aligned} G^{>}(t, \vec{x}; t', \vec{y}) &= \langle 0(t_0) | \varphi(t, \vec{x}) \varphi(t', \vec{y}) | 0(t_0) \rangle \\ G^{<}(t, \vec{x}; t', \vec{y}) &= \langle 0(t_0) | \varphi(t', \vec{y}) \varphi(t, \vec{x}) | 0(t_0) \rangle, \end{aligned} \tag{29}$$

where t_0 is later taken to be the far past, so that the vacuum corresponds to the Minkowski vacuum. In momentum representation, the Wightman function takes form of

$$G_k^{>}(t, t') = \varphi_k(t) \varphi_k^*(t'), \qquad G_k^{<}(t, t') = \varphi_k^*(t) \varphi_k(t'). \tag{30}$$

In conformal time, the leading behavior of the Wightman functions in the slow-roll limit is thus

$$\begin{aligned} G_k^{>}(t, t') &= \frac{1}{2} H^2 \frac{1}{k^3} (1 + ik\eta)(1 - ik\eta') e^{-ik(\eta - \eta')} + \cdots \\ G_k^{<}(t, t') &= \frac{1}{2} H^2 \frac{1}{k^3} (1 - ik\eta)(1 + ik\eta') e^{ik(\eta - \eta')} + \cdots. \end{aligned} \tag{31}$$

These Wightman functions are necessary to compute the expectation value of an operator $Q(t)$ using *in-in formalism* [18][19],

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \bar{T} \left(e^{i \int_{-\infty(1-i\epsilon)}^{t} dt' H_I(t')} \right) \mathcal{O}^I(t) T \left(e^{-i \int_{-\infty(1+i\epsilon)}^{t} dt' H_I(t')} \right) | 0 \rangle, \tag{32}$$

where $|0\rangle$ is the vacuum in the far past, ($\bar{T}$-)T is the (anti-)time ordering symbol, ϵ is a small parameter to project the interacting vacuum to the true vacuum as in usual sense of quantum field theory. The operator $\mathcal{O}(t)$ in this work will be the $\varphi(t, \vec{x}) \varphi(t', \vec{y})$.

Wick contractions involving these operators will then produce time derivatives of the Wightman functions as well. As a convention, let the time derivatives always be implicitly acting on the second of the arguments of the Wightman functions,

$$\dot{G}_k^{>}(t,t') \equiv \frac{d}{dt'} G_k^{>}(t,t') = \frac{d\eta'}{dt'}\frac{d}{d\eta'} G_k^{>}(t,t'). \tag{33}$$

The leading part of the derivative of a Wightman function is then

$$\begin{aligned} \dot{G}_k^{>}(t,t') &= -\frac{1}{2}H^3\frac{\eta'^2}{k}(1+ik\eta)e^{-ik(\eta-\eta')} + \cdots , \\ \dot{G}_k^{<}(t,t') &= -\frac{1}{2}H^3\frac{\eta'^2}{k}(1-ik\eta)e^{+ik(\eta-\eta')} + \cdots . \end{aligned} \tag{34}$$

Finally, the fluctuations of interest for the inhomogeneities of the universe are those that have been stretched beyond the Hubble horizon by the end of inflation. This corresponds to the $-k\eta \to 0$ limit. In this limit, the Wightman functions and their derivatives become

$$\begin{aligned} G_k^{>}(0,\eta') &\approx \frac{1}{2}H^2\frac{1}{k^3}(1-ik\eta')e^{+ik\eta'} \quad \dot{G}_k^{>}(0,\eta') \approx -\frac{1}{2}H^3\frac{\eta'^2}{k}e^{+ik\eta'}, \\ G_k^{<}(0,\eta') &\approx \frac{1}{2}H^2\frac{1}{k^3}(1+ik\eta')e^{-ik\eta'} \quad \dot{G}_k^{<}(0,\eta') \approx -\frac{1}{2}H^3\frac{\eta'^2}{k}e^{-ik\eta'}. \end{aligned} \tag{35}$$

These expressions are used in the evaluation of the expectation values of the operators necessary to obtain the 2-point function.

There are now two types of fields, $\varphi^{+}(t,\vec{x})$ and $\varphi^{-}(t,\vec{x})$ that corresponds to the field in forward and backward time integration respectively. Thus, there are four possible Wick contractions (the contraction is denoted by overbar),

$$\begin{aligned} \overline{\varphi^{\pm}(t,\vec{x})\varphi^{\pm}}(t',\vec{y}) &= \langle 0(t_0)|T\left(\varphi^{\pm}(t,\vec{x})\varphi^{\pm}(t',\vec{y})\right)|0(t_0)\rangle \\ &= G^{\pm\pm}(t,\vec{x};t',\vec{y}). \end{aligned} \tag{36}$$

By obtaining the expectation values of time-ordered pairs of fields, it is possible to express these propagators in terms of two Wightman functions,

$$\begin{aligned} G^{++}(t,\vec{x};t',\vec{y}) &= \Theta(t-t')G^{>}(t,\vec{x};t',\vec{y}) + \Theta(t'-t)G^{<}(t,\vec{x};t',\vec{y}), \\ G^{+-}(t,\vec{x};t',\vec{y}) &= G^{<}(t,\vec{x};t',\vec{y}), \\ G^{-+}(t,\vec{x};t',\vec{y}) &= G^{>}(t,\vec{x};t',\vec{y}), \\ G^{--}(t,\vec{x};t',\vec{y}) &= \Theta(t'-t)G^{>}(t,\vec{x};t',\vec{y}) + \Theta(t-t')G^{<}(t,\vec{x};t',\vec{y}). \end{aligned} \tag{37}$$

For the calculation performed in this study, we set the initial time as the infinite past, denoted as $t_0 \to -\infty$. In this limit, the interacting vacuum asymptotically transitions into the free vacuum state $|0(-\infty)\rangle \equiv |0\rangle$, which is commonly referred to as the Bunch-Davies state [20].

When the interactions are small, this expectation value can be evaluated perturbatively. The exponential appearing in (32) can be expanded in a Taylor series,

$$\begin{aligned}
&\langle 0(t)|\mathcal{O}(t)|0(t)\rangle \\
&= \langle 0(t)|T\left(\mathcal{O}^{+}(t)e^{-i\int_{-\infty}^{t}dt'\left[H_I^{+}(t')-H_I^{-}(t')\right]}\right)|0(t)\rangle \\
&= \langle 0(t_0)|\mathcal{O}^{+}(t)|0(t_0)\rangle \\
&\quad - i\int_{t_0}^{\infty}dt'\ \langle 0(t_0)|T(\mathcal{O}^{+}(t)[H_I^{+}(t') - H_I^{-}(t')])|0(t_0)\rangle + \cdots.
\end{aligned} \tag{38}$$

The zeroth order expansion is simply the tree-level two-point correlation function that can be expressed in terms of the zeroth-order power spectrum $P_{\phi,0}$ as

$$\begin{aligned}
&\langle 0(t)|\varphi(t,\vec{x})\varphi(t,\vec{y})|0(t)\rangle_0 \equiv \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{2\pi^2}{k^3} P_{\phi,0}(\eta), \\
&P_{\phi,0}(\eta) = \left(\frac{H}{2\pi}\right)^2.
\end{aligned} \tag{39}$$

The next order is what gives the correction to $P_{\phi,0}$, that is,

$$\begin{aligned}
&\langle 0(t)|\varphi(t,\vec{x})\varphi(t,\vec{y})|0(t)\rangle_1 \\
&= -i\int_{t_0}^{\infty}dt'\ \langle 0(t_0)|T(\varphi^{+}(t,\vec{x})\varphi^{+}(t,\vec{y})[H_I^{+}(t') - H_I^{-}(t')])|0(t_0)\rangle.
\end{aligned} \tag{40}$$

The calculation can be done separately, term by term, for the interaction Hamiltonian

$$H_I = -\mathcal{L}_{K,2}, \tag{41}$$

we identify terms in the quadratic action in Eq. (21) that differ from the free theory in the flat spatial section as interaction terms which are slow-roll suppressed, so that it can be evaluated perturbatively. Additionally, under the sub-curvature approximation, we can approximate $\sqrt{\gamma} = 1$.

The curvature terms in the Lagrangian as interaction terms that are suppressed by slow-roll parameters are

$$\begin{aligned}
&H_I = \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3 \\
&\mathcal{O}_1 = 3a^3K^2\chi^2, \qquad \mathcal{O}_2 = -a^3K\partial_i\chi\partial^i\chi, \qquad \mathcal{O}_3 = -\frac{3}{2}aK\varphi^2.
\end{aligned} \tag{42}$$

Now we can calculate the expectation values of the operators involved. The result is

$$\langle 0(t)|\varphi(t,\vec{x})\varphi(t,\vec{y})|0(t)\rangle_1 = \langle \mathcal{O}_1\rangle + \langle \mathcal{O}_2\rangle + \langle \mathcal{O}_3\rangle, \tag{43}$$
$$\langle \mathcal{O}_1\rangle = -\frac{3}{4}\epsilon H^2 K^2 \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{(-k^2+3K)^2}\frac{1}{k^3},$$
$$\langle \mathcal{O}_2\rangle = \frac{1}{4}\epsilon H^2 K \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{(-k^2+3K)^2}\frac{1}{k},$$
$$\langle \mathcal{O}_3\rangle = \frac{9}{4} H^2 K \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{k^5}.$$

The power spectrum of the scalar perturbation up to the first-order expansion becomes

$$P_\varphi = \left(\frac{H}{2\pi}\right)^2 \left(1 + \frac{1}{2}K\left(\frac{9}{k^2} + \frac{\epsilon}{k^2 - 3K}\right)\right). \tag{44}$$

We can relate this to the power spectrum of the curvature perturbation ζ to a linear order,

$$P_\zeta = \left(\frac{H}{\dot{\phi}}\right)^2 P_\phi\Big|_{k=aH},$$
$$P_\zeta = \left(\frac{1}{8\pi^2}\frac{1}{\epsilon}\frac{H^4}{M_{\mathrm{Pl}}}\right)\left(1 + \frac{1}{2}K\left(\frac{9}{k^2} + \frac{\epsilon}{k^2 - 3K}\right)\right). \tag{45}$$

The correction introduces momentum dependences that is absent in flat space power spectrum in Eq. (39), which is scale-invariant. Note that the corrections can be enhanced when the mode k is comparable to either the spatial curvature K or the slow roll parameter ϵ.

4 Conclusion

In this paper we have calculated the corrections to the inflationary power spectrum of primordial curvature perturbations originating from small initial spatial curvature prior to inflation. By employing the Friedman-Lemaître-Robertson-Walker (FLRW) model with spatial curvature and single-field slow-roll inflation, we have demonstrated that the presence of spatial curvature introduces additional terms to the second-order action of perturbations, proportional to the spatial curvature itself. Specifically, we focused on investigating spatial curvature of comparable magnitude to the slow-roll parameter.

This result adds a possible observable in which the primordial curvature of the universe can be detected, as well as other quantities such as the inflationary bispectrum and trispectrum. The correction breaks the approximate scale-invariant in flat space power spectrum. Through a perturbative evaluation around the standard free part of the action in flat space and the application of the Bunch-Davies boundary condition to normalize the mode function, we determined that the corrections to the power spectrum exhibit a proportional relationship with the spatial curvature, particularly for short-wavelength modes. The corrections can be enhanced when the largest observable scale is comparable to the curvature scale, as shown in Eq. (45). Although this correction currently lies beyond the detection limit of current microwave background experiments, future observational techniques could allow for greater accuracy in determining the parameter range for the spatial curvature [21]-[23].

For future investigations, it would be interesting to calculate the power spectrum more rigorously by relaxing the assumption of small initial curvature and considering all terms in Eq. (19). This would require a mode function that is better suited to describing the curved spatial section. In addition, if feasible, it would also be useful to analytically solve the equation of motion to get the power spectrum. Furthermore, this power spectrum can be numerically evolved into the temperature anisotropy spectrum of the cosmic microwave background (CMB), which can be directly compared to the Planck CMB data.

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