



LSMPS Method for Solving Steady Heat Conduction Equation in Orthotropic Materials

Michael Agung Nugroho*, Pramudita Satria Palar, Gea Fardias Mu'min & Lavi Rizki Zuhail

Faculty of Mechanical and Aerospace Engineering, Institut Teknologi Bandung,
Jalan Ganesa 10, Bandung 40132, Indonesia
*Email: michael.agung67@gmail.com

Abstract. Heat conduction in orthotropic materials has been an important topic in engineering. In this study, LSMPS method was used to solve steady heat conduction problem for orthotropic medium. Two methods were proposed in this paper, one using coordinate transformation and one without coordinate transformation. The numerical results were then compared to the analytical solution. The results from both methods showed good agreement with the analytical solution. The coordinate transformation method had slightly better accuracy, though it also took slightly longer computation time. However, the direct method, without coordinate transformation is more practical when dealing with non-orthogonality and non-homogeneity over the domain. In general, the proposed methods have been able to solve steady heat conduction problem for orthotropic medium.

Keywords: *anisotropic materials; heat conduction; LSMPS; meshless method; orthotropic.*

1 Introduction

Heat conduction in orthotropic materials is an important topic in the field of engineering and material science. These materials exhibit varying thermal conductivity along different directions due to their anisotropic nature. Thus, accurate modeling of heat transfer in orthotropic materials is critical for the design and optimization of various engineering applications such as aerospace, automotive, and biomedical devices. Standard heat conduction textbooks, such as the work of Carslaw and Jaeger [1] and Ozisik [2], have some portions of their contents talking about anisotropic heat conduction.

Several methods have been developed to solve the physics of anisotropic heat conduction. Earlier studies tried to solve anisotropic heat conduction problems analytically. Chang *et al.* [3] solved anisotropic heat conduction analytically using Green's functions, while Ma and Chang [4] used Fourier transform to obtain explicit solution for anisotropic multi-layered media. Recent studies

solved anisotropic heat conduction phenomena by means of numerical simulation. Standard numerical methods such as Finite Difference Method (FDM) [5-6], Finite Volume Method (FVM) [7-8], and Finite Element Method (FEM) [9-10] were used to solve anisotropic heat conduction phenomena. Sladek *et al.* [11] solved nonhomogeneous anisotropic heat conduction problem based on Petrov-Galerkin meshless method. More recent studies saw several other meshless methods [12-16] were used to solve anisotropic heat conduction problems.

In this paper, we present a study on the use of Least Square Moving Particle Semi-Implicit (LSMPS) method to simulate steady heat conduction in orthotropic materials. LSMPS is a high-order accurate meshfree discretization method developed by Tamai and Koshizuka [17]. LSMPS method satisfies the consistency, polynomial completeness, and reproducing conditions [17], which most other weakform meshless methods are lacking. LSMPS has been used to solve thermal conduction problem by Tanaka *et al.* [18]. Tanaka *et al.* [18] used LSMPS discretization to ensure the satisfaction of the conservation of energy. Another advantage of using LSMPS method is its ability to be used with multiresolution model [19] and spherical particle model [18], which can reduce the computational cost.

This study focuses on the implementation of the LSMPS method and its accuracy in predicting the temperature distribution in orthotropic materials. We present the numerical implementation of the LSMPS method for simulating steady heat conduction in orthotropic materials. We also provide numerical results and comparisons with analytical solutions to demonstrate the accuracy of the method.

2 Problem Statement

In this study, a thin rectangular film, assumed as 2-dimensional, is considered. The film is made of quartz with thermal conductivity of $k_{11} = 6.5 \text{ W/m}^2\text{K}$ and $k_{22} = 11.3 \text{ W/m}^2\text{K}$. The dimension of the film is $[0,10]\text{cm} \times [0,10]\text{cm}$. It is assumed that there is no heat source, and the film has reached steady condition. The temperature of the boundary is defined as a function of the position.

$$T(x, 0) = T(0, y) = 0 \quad (1)$$

$$T(x, 10) = 10x(10 - x) \quad (2)$$

$$T(10, y) = \frac{100y(10 - y)}{y + 3} \quad (3)$$

3 Methodology

The general equation governing the steady state heat conduction phenomena is given in Eq. (4).

$$\int \vec{\nabla} \cdot (\mathbf{k}\vec{\nabla}T) dV + \oint \vec{q} \cdot d\vec{S} = 0 \quad (4)$$

Where \mathbf{k} denotes the conductivity tensor, T denotes the temperature, \vec{q} denotes the heat flux from the external region, V denotes the volume, and \vec{S} denotes the normal vector of the surface.

Assuming that there is no external heat flux, Eq. (4) can be rewritten by applying Gauss' Theorem to the first term of the left-hand side.

$$\oint (\mathbf{k}\vec{\nabla}T) \cdot d\vec{S} = 0 \quad (5)$$

For each particle, the governing equation in Eq. (5) is discretized following the work of Tanaka *et al.* [18].

$$\sum_{j \neq i} \left[\frac{1}{2} (\mathbf{k}\vec{\nabla}T)_{ij} (\vec{S}_{ij} - \vec{S}_{ji}) \right] + (\mathbf{k}\vec{\nabla}T)_i \vec{B}_i = 0 \quad (6)$$

where $(\mathbf{k}\vec{\nabla}T)_i$, $(\mathbf{k}\vec{\nabla}T)_{ij}$, \vec{S}_{ij} , and \vec{B}_i are the temperature gradient, inter-particles heat flux, surface vector, and boundary vector respectively.

Temperature gradient, surface vector, and boundary vector are calculated using standard LSMPS method. On the other hand, the inter-particles heat flux $(\mathbf{k}\vec{\nabla}T)_{ij}$ in Eq. (6) is calculated by giving weight α_{ij} to the temperature gradient terms $(\mathbf{k}\vec{\nabla}T)_i$ and $(\mathbf{k}\vec{\nabla}T)_j$.

$$(\mathbf{k}\vec{\nabla}T)_{ij} = \alpha_{ij} (\mathbf{k}\vec{\nabla}T)_i + (1 - \alpha_{ij}) (\mathbf{k}\vec{\nabla}T)_j \quad (7)$$

Eq. (6) and (7) ensure the satisfaction of the first law of thermodynamics, as the heat flux leaving particle i is the same as heat flux entering particle j , or vice versa, for every pair of neighboring particles.

The thermal conductivity tensor \mathbf{k} for an orthotropic medium can be written as in Eq. (8).

$$\mathbf{k} = \begin{bmatrix} k_{11} & 0 \\ 0 & k_{22} \end{bmatrix} \quad (8)$$

Where k_{11} and k_{22} denote the thermal conductivity in both principal axes.

Due to this anisotropy, some modifications are needed to solve the problem of heat conduction. In this paper, 2 methods are developed to solve the problem of heat conduction.

3.1 Method 1

The term $\mathbf{k}\vec{\nabla}T$, for each particle, can be expanded by substituting the conductivity matrix \mathbf{k} and the nabla operator $\vec{\nabla}$. The partial derivatives itself are calculated by LSMPS method.

$$\mathbf{k}\vec{\nabla}T = k_{11} \frac{\partial T}{\partial x} \hat{i} + k_{22} \frac{\partial T}{\partial y} \hat{j} \quad (9)$$

The value of α_{ij} in Eq. (7) is determined by solving the following equation.

$$\alpha_{ij}^* (\mathbf{k}\vec{\nabla}T)_i + (1 - \alpha_{ij}^*) (\mathbf{k}\vec{\nabla}T)_j = \frac{T_j - T_i}{|\vec{x}_j - \vec{x}_i|} \mathbf{k} \cdot \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|} \quad (10)$$

Where α_{ij}^* is the temporary value for α_{ij} . If the value of α_{ij}^* is between 0 and 1, then $\alpha_{ij} = \alpha_{ij}^*$. If α_{ij}^* is less than 0 or more than 1, the value of α_{ij} is assumed to be 0 and 1 respectively. Otherwise, it is assumed that $\alpha_{ij} = 0.5$.

3.2 Method 2

Heat conduction equation for an orthotropic medium can be transformed to a standard isotropic heat conduction equation by coordinate transformation, as given in the work of Ozisik [2]. The isotropic conductivity \bar{k} and the coordinate transformation matrix are defined as in Eq. (11) and Eq. (12).

$$\bar{k} = \sqrt{k_{11}k_{22}} \quad (11)$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sqrt{\bar{k}/k_1} & 0 \\ 0 & \sqrt{\bar{k}/k_2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (12)$$

By using these transformations, the term $\mathbf{k}\vec{\nabla}T$ can be written in the transformed coordinate as follows.

$$\mathbf{k}\vec{\nabla}T = \bar{k} \frac{\partial T}{\partial X} \hat{i} + \bar{k} \frac{\partial T}{\partial Y} \hat{j} \quad (13)$$

As in Method 1, temporary variable α_{ij}^* is introduced to calculate the value of α_{ij} .

$$\alpha_{ij}^* (\mathbf{k}\vec{\nabla}T)_i + (1 - \alpha_{ij}^*) (\mathbf{k}\vec{\nabla}T)_j = \bar{k} \frac{T_j - T_i}{|\vec{X}_j - \vec{X}_i|} \frac{\vec{X}_j - \vec{X}_i}{|\vec{X}_j - \vec{X}_i|} \quad (14)$$

As was the case in Method 1, if the value of α_{ij}^* is between 0 and 1, then $\alpha_{ij} = \alpha_{ij}^*$. If α_{ij}^* is less than 0 or more than 1, the value of α_{ij} is assumed to be 0 and 1 respectively. Otherwise, it is assumed that $\alpha_{ij} = 0.5$.

3.3 Analytical Solution

The method for calculating the solution of the problem analytically follows the work of Ozisik [2].

Coordinate transformation defined in Eq. (11) and Eq. (12) is also used for analytical solution. By transforming the coordinate, the governing equation of steady heat conduction equation can be written as in Eq. (15).

$$\bar{k} \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right) = 0 \quad (15)$$

Eq. (15) can be solved by using variable separation method. As there are 2 non-homogeneous boundary conditions, i.e., Eq. (2) and Eq. (3), the analytical solution also used superposition method.

For the sake of simplicity, let's define L and W be the length and the width of the transformed rectangular, i.e., $0 \leq X \leq L$ and $0 \leq Y \leq W$. Also, let $\alpha(X)$ and $\beta(Y)$ be the transformed boundary conditions given in Eq. (2) and Eq. (3), respectively.

In short, the analytical solution for the proposed problem is given in Eq. (16).

$$T(X, Y) = \sum_{n=1}^{\infty} [C_n \sin(\lambda_{1,n}X) \sinh(\lambda_{1,n}Y) + D_n \sin(\lambda_{2,n}Y) \sinh(\lambda_{2,n}X)] \quad (16)$$

Where $\lambda_{1,n}$, $\lambda_{2,n}$, C_n , and D_n are the frequencies and coefficients of the Fourier series.

$$\lambda_{1,n} = \frac{n\pi}{L} \quad (17)$$

$$\lambda_{2,n} = \frac{n\pi}{W} \quad (18)$$

$$C_n = \frac{2 \sum_{k=1}^m \alpha \left(\frac{(k-0.5)L}{m} \right) \sin \left(\lambda_{1,n} \frac{(k-0.5)L}{m} \right)}{m \sinh(\lambda_{1,n}W)} \quad (19)$$

$$D_n = \frac{2 \sum_{k=1}^m \beta \left(\frac{(k-0.5)W}{m} \right) \sin \left(\lambda_{2,n} \frac{(k-0.5)W}{m} \right)}{m \sinh(\lambda_{2,n}L)} \quad (20)$$

4 Results and Discussion

The temperature for the 9801 (99×99) internal nodes is calculated by using both numerical and analytical methods. Both numerical results are compared to the analytical solution and its errors are quantified by L_2 norm error.

$$\frac{\|\hat{x} - x\|}{\|x\|} = \sqrt{\frac{\sum_i \|\hat{x}_i - x_i\|^2}{\sum_i \|x_i\|^2}} \quad (21)$$

The L_2 norm error for the first method is 0.0028, while it is 0.0013 for the second method. Method 1 takes about 42ms for each iteration, while each iteration takes 46ms for method 2. When comparing the total thermal energy across the whole domain, both methods are very similar. Method 1 has an error of 0.0405%, while the second method has an error of 0.0403%. However, the total energy of the first method is higher than the analytical solution, while the second method results in lower total energy than the analytical solution.

The contour plot results are presented in Figure 1. In addition, the isothermal lines with an increment of 10° each are also added to the contour plots.

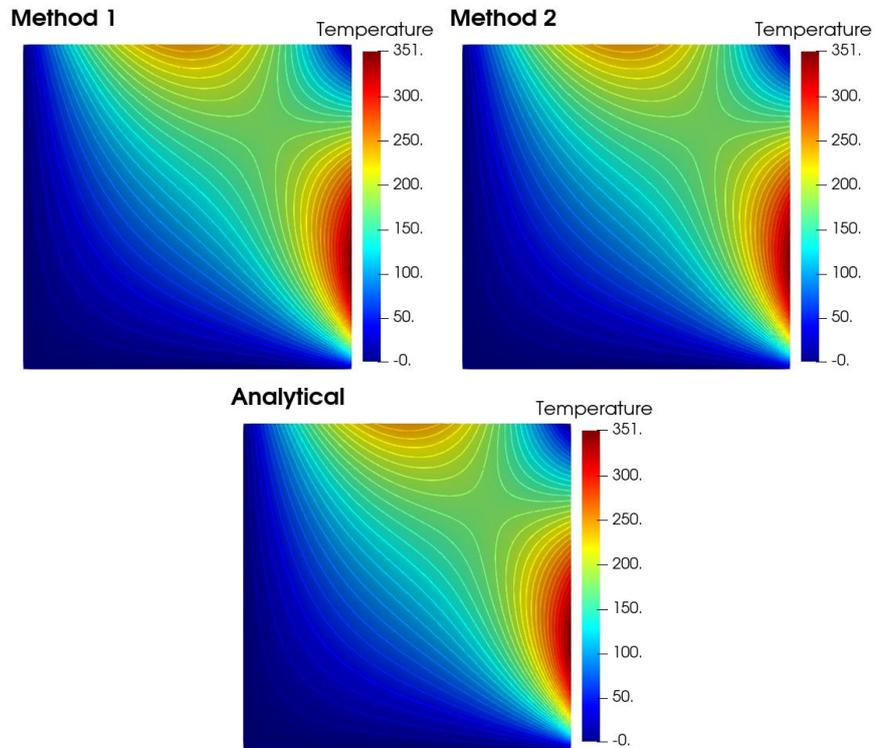


Figure 1 Contour plots of the steady state temperature solution

The value of L_2 norm error and total thermal energy error for both methods, coupled with the contour plots agreement in Figure 1, show that both numerical methods are indeed able to solve the steady heat conduction equation for an orthotropic medium. However, it must also be noted that Simpson method was used to calculate the definite integral in the analytical solution, therefore the analytical result itself is not completely analytical.

For the sake of comparison, steady temperature results are monitored at $y = 3$ line and presented in Figure 2. Figure 2 shows that both numerical results are very similar to the analytical solution. In general, the result from the first method is slightly higher than the analytical solution, while the second method results in lower temperature distribution compared to the analytical solution.

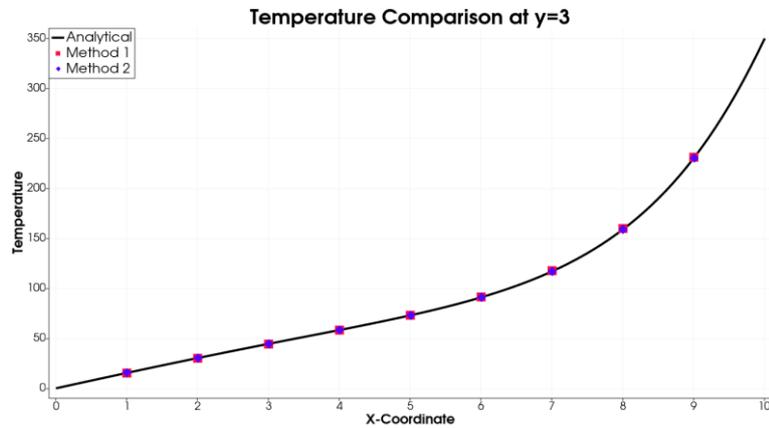


Figure 2 Temperature comparison at $y = 3$

Comparing the L_2 norm error, the second method results in lower L_2 norm error, which means that the accuracy of the second method is higher than that of method 1. This is indeed true for a particularly simple simulation, as was the case in this simulation. However, the second method may run into several difficulties when it is used to simulate more complex simulation. The coordinate transformation used in the second method, as given in Eq. (11) and Eq. (12), is solely based on the conductivity tensor \mathbf{k} , which in this simulation is assumed to be constant and orthogonal throughout the entire simulation domain. If the conductivity matrix is non-orthogonal, i.e., the off-diagonal components are non-zero, much more complex transformation matrix than what is given in Eq. (12) may need to be used. Finding the correct transformation matrix for a non-orthogonal conductivity matrix may prove to be difficult, but it is still possible.

This is not the case, however, if the conductivity is non-homogeneous along the domain, i.e., the conductivity at one location is different to that of other locations. One example of this case is when the conductivity of the material depends on the temperature. Another example is when the domain simply consists of more than one material. In this case, finding the correct transformation matrix may not be possible at all.

On the other hand, solving the non-orthogonality of the conductivity matrix in the first method can be achieved by adding the off-diagonal terms in the Eq. (9). The final equation to be solved, which is the Eq. (10), remains the same whether the conductivity matrix is non-orthogonal or non-homogeneous. Therefore, the first method is more practical to be used in simulating heat conduction phenomena, albeit the slightly lower accuracy. Note that this has not factored in

that the second method takes roughly 10% more computational time than the first method, which makes the first method much more plausible.

5 Conclusion

Steady state heat conduction equation for two-dimensional orthotropic rectangular medium is solved both numerically and analytically. The analytical solution uses coordinate transformation to obtain simple Laplace equation, and then uses the method of variable separation. On the other hand, the numerical solution is based on LSMPS discretization. In this study, two different numerical methods were introduced, one using coordinate transformation and one without using so. Compared to the analytical solution, the result of using the direct, without coordinate transformation method has slightly lower accuracy than the transformation method. However, the direct method is more practical when dealing with non-orthogonality and non-homogeneity over the domain. Nevertheless, the numerical simulation results show that both methods have good agreement with the analytical solution and are indeed able to solve steady two-dimensional heat conduction for orthotropic medium problems.

6 Nomenclature

α_{ij}	=	Weight for particle i and j interaction
α_{ij}^*	=	Temporary value of α_{ij}
$\alpha(X)$	=	Boundary condition for the top surface in \vec{X} coordinate
\vec{B}_i	=	Boundary vector of particle i
$\beta(Y)$	=	Boundary condition for the right surface in \vec{X} coordinate
C_n	=	Fourier series' coefficient
D_n	=	Fourier series' coefficient
\mathbf{k}	=	Conductivity matrix
k_{11}	=	Conductivity in X-direction
k_{22}	=	Conductivity in Y-direction
\bar{k}	=	Transformed conductivity
$(\mathbf{k}\vec{\nabla}T)_i$	=	Temperature gradient of particle i
$(\mathbf{k}\vec{\nabla}T)_{ij}$	=	Heat flux between particle i and j
L	=	Transformed domain length in the X -direction
$\lambda_{1,n}$	=	Fourier series' frequency
$\lambda_{2,n}$	=	Fourier series' frequency
\vec{q}	=	External heat flux
\vec{S}	=	Surface vector to external region
\vec{S}_{ij}	=	Surface vector between particle i and j

T	=	Temperature
W	=	Transformed domain length in the Y -direction
\vec{x}_i	=	Position of particle i in (x, y) coordinates
\vec{X}_i	=	Position of particle i in (X, Y) coordinates

References

- [1] Carslaw, H.S. & Jaeger, J.C., *Conduction of Heat in Solids*, Oxford University Press, 1959.
- [2] Ozisik, M.N., *Heat Conduction*, Wiley, 1993.
- [3] Chang, Y.P., Kang, C.S. & Chen, D.J., *The Use of Fundamental Green's Functions for the Solution of Problems of Heat Conduction in Anisotropic Media*, International Journal of Heat and Mass Transfer, **16**(10), pp. 1905-1918, October 1973.
- [4] Ma, C.C. & Chang, S.W., *Analytical Exact Solutions of Heat Conduction Problems for Anisotropic Multi-layered Media*. International Journal of Heat and Mass Transfer, **47**(8), pp. 1643-1655, April 2004.
- [5] Sameti, M., Astaraiie, F.R., Pourfayaz, F. & Kasaeian, A., *Analytical and FDM Solutions for Anisotropic Heat Conduction in an Orthotropic Rectangular*, American Journal of Numerical Analysis, **2**(2), pp. 65-68, March 2014.
- [6] Yan, G., Qingsong, H., Chuanzeng, Z. & Xiaoqiao, H., *The Generalized Finite Difference Method for Long-time Transient Heat Conduction in 3D Anisotropic Composite Materials*, Applied Mathematical Modelling, **71**, pp. 316-330, July 2019.
- [7] Van Es, B., Koren, B. & de Blank, H.J., *Finite-volume Scheme for Anisotropic Diffusion*, Journal of Computational Physics, **306**, pp. 422-442, February 2016.
- [8] Prestini, D., Filippini, G., Zdanski, P.S.B. & Vaz Jr, M., *Fundamental Approach to Anisotropic Heat Conduction using the Element-Based Finite Volume Method*, International Journal of Computation and Methodology, **71**(4), pp. 327-345, April 2017.
- [9] Bazyar, M.H. & Talebi, A., *Scaled Boundary Finite-element Method for Solving Non-Homogeneous Anisotropic Heat Conduction Problems*, Applied Mathematical Modelling, **39**(23-24), pp. 7583-7599, December 2015.
- [10] Annasabi, Z. & Erchiqui, F. *3D Hybrid Finite Elements for Anisotropic Heat Conduction in a Multi-material with Multiple Orientations of the Thermal Conductivity Tensors*, International Journal of Heat and Mass Transfer, **158**, September 2020.
- [11] Sladek, J., Sladek, V., & Atluri, S. N., *Meshless Local Petrov-Galerkin Method for Heat Conduction Problem in an Anisotropic Medium*,

- Computer Modeling in Engineering and Sciences, **6**(3), pp. 309-318, September 2004.
- [12] Hidayat, M.I.P, *Meshless Local B-spline Collocation Method for Heterogeneous Heat Conduction Problems*, Engineering Analysis with Boundary Elements, **101**, pp. 76-88, April 2019.
- [13] Qiu, L., Wang, F. & Lin, J., *A Meshless Singular Boundary Method for Transient Heat Conduction Problems in Layered Materials*, Computers & Mathematics with Applications, **78**(11), pp. 3544-3562, December 2019.
- [14] Ud Din, Z., Ahsan, M., Ahmad, M., Khan, W., Mahmoud, E.E. & Abdel-Aty, A-H., *Meshless Analysis of Nonlocal Boundary Value Problems in Anisotropic and Inhomogeneous Media*, Mathematics, **8**(11), November 2020.
- [15] Wang, C., Wang, F. & Gong, Y., *Analysis of 2D Heat Conduction in Nonlinear Functionally Graded Materials using a Local Semi-Analytical Meshless Method*, AIMS Mathematics, **6**(11), pp. 12599-12618, September 2021.
- [16] Chen, H. & Liu, D., *Formulation of a Nonlocal Discrete Model for Anisotropic Heat Conduction Problems*, International Journal of Thermal Sciences, **182**(1), December 2022.
- [17] Tamai, T. & Koshizuka, S., *Least Square Moving Particle Semi-Implicit Method*, Computational Particle Mechanics, **1**(3), pp. 277-305, September 2014.
- [18] Tanaka, M., Cardoso, R. & Bahai, H. *Modification of the LSMPS Method for the Conservation of the Thermal Energy in Laser Irradiation Processes*, International Journal for Numerical Methods in Engineering, **117**(2), pp. 161-187, September 2018.
- [19] Tanaka, M., Cardoso, R. & Bahai, H., *Multi-resolution MPS Method*, Journal of Computational Physics, **359**, pp. 106-136, 2018.